Show Your Work! Point values are in square brackets. There are 40 points possible. If you need more space, use the backs of these pages.

1. Suppose \( n \in \mathbb{Z} \). Consider the following two predicates:

   \( P(n) : n^2 + n - 2 = 0 \) \hspace{1cm} \( Q(n) : n^2 + 2n - 3 = 0 \).

   Find \( \{ n \in \mathbb{Z} : P(n) \lor Q(n) \} \). [2 points]

\[
n^2 + n - 2 = (n + 2)(n - 1) \hspace{1cm} \text{and} \hspace{1cm} n^2 + 2n - 3 = (n + 3)(n - 1).
\]

So \( P(n) \) is true when \( n = -2 \) or \( n = 1 \),

and \( Q(n) \) is true when \( n = -3 \) or \( n = 1 \).

So \( \{ n \in \mathbb{Z} : P(n) \lor Q(n) \} = \{-3, -2, 1\} \).

2. Suppose that \( I \) is the set of irrational numbers, \( \mathbb{Q} \) is the set of rational numbers, and \( \mathbb{R} \) is the set of real numbers. If \( \mathbb{R} \) is the universal set, find

   (a) \( \mathbb{Q} - I = \mathbb{Q} \)

   (b) \( \mathbb{Q} \oplus I = (\mathbb{Q} - I) \cup (I - \mathbb{Q}) = \mathbb{Q} \cup I = \mathbb{R} \)

   (c) \( \mathbb{Q} \cap I = \emptyset \)

[3]
3. Consider the set 
\[ A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \emptyset\}. \]
(a) Is \(\emptyset \in A\)? Yes
(b) Is \(\emptyset \subset A\)? Yes
(c) What is the cardinality of \(A\)? 3. Repeated elements don’t affect cardinality.

4. Find the power set of \(\{2, \{1\}\}\). \[\mathcal{P}(\{2, \{1\}\}) = \{\emptyset, \{2\}, \{\{1\}\}, \{2, \{1\}\}\}\]
5. In order to prove the statement “\( n \) is even if and only if \( n + 2 \) is even,” Sally is going to prove two implications. What two implications should Sally prove? You do not need to prove anything for this problem: just state the implications that Sally should prove. [2]

Sally should prove that if \( n \) is even then \( n + 2 \) is even. She should also prove that if \( n + 2 \) is even then \( n \) is even.

6. Suppose \( A, B, \) and \( C \) are sets. Prove or disprove:

\[
A \cup (B - C) = (A \cup B) - (A \cup C).
\]

Note that you should not assume the existence of a universal set. [2]

This is false: A Venn diagram shows clearly that \( A \cup (B - C) \neq (A \cup B) - (A \cup C) \). However, Venn diagrams don’t constitute proofs or disproofs. So we need an example. Consider

\[
A = \{1\}, \quad B = \{1, 2\}, \quad \text{and} \quad C = \emptyset.
\]

Then \( B - C = B \). So

\[
A \cup (B - C) = A \cup B = \{1\} \cup \{1, 2\} = \{1, 2\}.
\]

On the other hand,

\[
(A \cup B) - (A \cup C) = \{1, 2\} - \{1\} = \{2\}.
\]

So

\[
A \cup (B - C) \neq (A \cup B) - (A \cup C).
\]
7. If \( n \) is an odd integer, prove that there exists an integer \( m \) such that \( n = 4m + 1 \) or \( n = 4m + 3 \).

Suppose \( n \) is an odd integer. Then there is an integer \( k \) such that \( n = 2k + 1 \). We consider two cases: \( k \) is even and \( k \) is odd.

**Case 1.** If \( k \) is even, there is an integer \( m \) such that \( k = 2m \). So

\[
n = 2k + 1 = 2(2m) + 1 = 4m + 1,
\]

and in this case \( n \) can be written as \( 4m + 1 \).

**Case 2.** If \( k \) is odd, there is an integer \( m \) such that \( k = 2m + 1 \). So

\[
n = 2k + 1 = 2(2m + 1) + 1 = 4m + 3,
\]

and in this case \( n \) can be written as \( 4m + 3 \).

8. Suppose \( f : \mathbb{R} \to \mathbb{R} \) by \( f(x) = x^2 - 3 \).

(a) What is the codomain of \( f \)?

\( \mathbb{R} \), since by definition \( f : \mathbb{R} \to \mathbb{R} \).

(b) What is the range of \( f \)?

The square of a real number is \( \geq 0 \). So \( f(x) \geq 0 - 3 = -3 \), and the range is

\[
\{ y \in \mathbb{R} \mid y \geq -3 \} = [-3, \infty).
\]

(c) Is \( f \) one-to-one?

No. For example, \( f(-1) = -2 = f(1) \).
9. The relation $S$ is defined on $\mathbb{R}$ by $xSy$ if $x - y \geq 0$. State whether $S$ is

(a) Reflexive. Yes. If $x \in \mathbb{R}$, then $x - x = 0 \geq 0$. So $xSx$, for all $x \in \mathbb{R}$.

(b) Symmetric. No. For example, if $x = 1$ and $y = 0$, then $x - y = 1 - 0 = 1 \geq 0$. So $xSy$, but $y - x = 0 - 1 = -1 < 0$. So $ySx$, and $S$ is not symmetric.

(c) Transitive. Yes. Suppose $x$, $y$, and $z$ are real numbers such that $xSy$ and $ySz$. Then $x - y \geq 0$ and $y - z \geq 0$. So

$$ (x - y) + (y - z) \geq 0. $$

But then

$$ (x - y) + (y - z) = x - z \geq 0. $$

So $xSz$, and $S$ is transitive.

Give reasons for your answers. [3]

10. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are defined by $f(x) = 3x + 1$ and $g(x) = x^2 - 1$. Find

(a) $(f \circ g)(2) = f[g(2)] = f[4 - 1] = f(3) = 3 \cdot 3 + 1 = 10$

(b) $(g \circ f)(x) = g[f(x)] = g[3x + 1] = (3x + 1)^2 - 1 = 9x^2 + 6x$

Simplify your answers. [2]
11. Prove or disprove: if $x$ is a real number, then

$$[x]^2 = [x^2].$$

[2]

This is false. For example, suppose $x = 3/2$. Then

$$[x]^2 = [1.5]^2 = 1^2 = 1.$$  

But

$$[x^2] = [9/4] = [2.25] = 2.$$  

12. Suppose that $g : \mathbb{Z} \rightarrow \mathbb{Z}$ by $g(x) = 2x - 5$. Determine whether $g$ has an inverse. Explain your answer. [2]

The function $g$ does not have an inverse. This follows from the theorem that a function $f : A \rightarrow B$ is invertible iff $f$ is both one-to-one and onto. The function $g$ is not onto. For example, $0 \in \mathbb{Z}$ but $0$ is not in the range of $g$. If it were, we would have

$$2x - 5 = 0 \Rightarrow 2x = 5 \Rightarrow x = 5/2,$$

and we would get that $5/2$ is an integer.
13. Suppose $P$, $Q$, and $R$ are statements. Suppose also that $(Q \lor R) \Rightarrow (\neg P)$ is false and $Q$ is false. What are the truth values of $P$ and $R$? [2]

An implication is false only when the hypothesis is true and the conclusion is false. So

\[ Q \lor R \text{ is true, and } \neg P \text{ is false.} \]

Since $Q$ is false by assumption, $Q \lor R$ can only be true if

$R$ is true.

Since $\neg P$ is false, $P$ is also true.

14. Negate each of the following statements. Your answer should be in idiomatic English, and it should not use the words “no” or “not.”

(a) $\exists x \in \mathbb{Q}, x^2 < 0$.
   Formally, the negation is $\forall x \in \mathbb{Q}, x^2 \geq 0$.
   In English, this might be “the square of every rational number is nonnegative.”

(b) $\forall x \in \mathbb{Q}, \exists y \in \mathbb{Q}, xy > 0$.
   Formally, the negation is $\exists x \in \mathbb{Q}, \forall y \in \mathbb{Q}, xy \leq 0$.
   In English, this might be, “there’s a rational number with the property that it’s product with any other rational number is nonpositive.”
15. Bob is trying to give a direct proof of the following result:

Suppose $x$ is a real number. If $2x + 1$ is rational, then $x$ is rational.

(a) What should Bob assume? Explain what his assumptions mean. [2]
(b) What does Bob need to prove? Explain what this means. [2]
(c) Write a direct proof of Bob’s result. [3]

Your answers must consist of complete sentences in clear English.

(a) Bob should assume that $x$ is a real number, and $2x + 1$ is rational. So Bob is assuming that there are two integers $a$ and $b$, with $b \neq 0$, such that $2x + 1 = a/b$.
(b) Bob needs to prove that $x$ is a rational number. This means that there are two integers $c$ and $d$, with $d \neq 0$, such that $x = c/d$.
(c) Since $2x + 1 = \frac{a}{b}$, we can use rational arithmetic to solve for $x$. First, subtracting 1 from both sides gives

$$2x = \frac{a}{b} - 1 = \frac{a - b}{b}.$$ 

Now, multiplying both sides by $1/2$ gives

$$x = \frac{1}{2} \cdot \frac{a - b}{b} = \frac{a - b}{2b}.$$ 

Since $a$ and $b$ are integers, $a - b$ is an integer. Also, since $b$ is a nonzero integer, $2b$ is a nonzero integer. So if we define $c = a - b$, and $d = 2b$, we see that $x = c/d$ is a rational number.